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A polynomial path-following interior point algorithm for general linear complementarity problems

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Abstract Linear Complementarity Problems ($LCPs$) belong to the class of $\mathbb{NP}$-complete problems. Therefore we cannot expect a polynomial time solution method for $LCPs$ without requiring some special property of the coefficient matrix. Our aim is to construct interior point algorithms which, according to the duality theorem in EP (Existentially Polynomial-time) form, in polynomial time either give a solution of the original problem or detects the lack of property $\mathcal{P}_*(\tilde{\kappa})$, with arbitrary large, but apriori fixed $\tilde{\kappa}$). In the latter case, the algorithms give a polynomial size certificate depending on parameter $\tilde{\kappa}$, the initial interior point and the input size of the LCP). We give the general idea of an EP-modification of interior point algorithms and adapt this modification to long-step path-following interior point algorithms.

Keywords Linear complementarity problem · Sufficient matrix · $\mathcal{P}_*$-matrix · Interior point method · Long-step method

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1 Introduction

Consider the *linear complementarity problem* (LCP) that aims to find vectors $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n$ that satisfy

$$-M\mathbf{x} + \mathbf{s} = \mathbf{q}, \quad \mathbf{xs} = \mathbf{0}, \quad \mathbf{x},\ \mathbf{s} \geq \mathbf{0}, \tag{1}$$

where $M \in \mathbb{R}^{n\times n}$ and $\mathbf{q} \in \mathbb{R}^n$, and the notation $\mathbf{xs}$ is used for the coordinatewise (Hadamard) product of the vectors $\mathbf{x}$ and $\mathbf{s}$.

The LCP belongs to the class of $\mathbb{NP}$-complete problems, since the feasibility problem of linear equations with binary variables can be described as an LCP [13]. Therefore we cannot expect an efficient (polynomial time) solution method for LCPs without requiring some special property of the matrix M.

There are several polynomial time algorithms for solving an LCP if the matrix M is a positive semidefinite matrix, see e.g. [11,12,17,21]. Furthermore, an LCP can be solved in polynomial time if the matrix M is a $\mathcal{P}_*(\kappa)$-matrix,[1] however, in this case the computational complexity of the algorithm depends on κ too (see e.g., [8,15,17]). Positive semidefiniteness of a matrix can be checked in strongly polynomial time [14], but no polynomial time algorithm is known for checking whether a matrix is $\mathcal{P}_*(\kappa)$ or not. The best known test for the $\mathcal{P}_*(\kappa)$ property, introduced by Väliaho [20], is not polynomial.

For applying an interior point method (IPM) to an LCP with a $\mathcal{P}_*(\kappa)$-matrix, we need an initial interior point (or use an infeasible IPM) and one need to know apriori the κ value of the matrix M. An initial interior point can be found by using an embedding model [18], but the apriori knowledge of κ is a too strong assumption. Potra and Liu [17] softened this assumption, they modified their IPM in such a way, that we need to know only the sufficiency of the matrix. However, this is still a condition, that cannot be verified in polynomial time. Consequently, there is a need to design such an algorithm, that can handle any LCP with an arbitrary matrix. Therefore, in this paper we give a general method how to modify interior point algorithms for $\mathcal{P}_*(\kappa)$-matrix LCPs, with the goal to process general LCPs in polynomial time. The new algorithms, in polynomial time either solve the LCP, or give a polynomial size certificate that the matrix M is not a $\mathcal{P}_*(\tilde{\kappa})$-matrix with arbitrary large, but apriori fixed $\tilde{\kappa}$. Polynomiality depends on the parameter $\tilde{\kappa}$, the initial interior point and the input size of the LCP. Throughout the paper, except in Section 4, we assume that a feasible interior point of the LCP is known.

Let now consider the *dual linear complementarity problem* ($DLCP$) [3]: find vectors $\mathbf{u}, \mathbf{z} \in \mathbb{R}^n$ which satisfy the constraints

$$\mathbf{u} + M^T\mathbf{z} = \mathbf{0}, \quad \mathbf{q}^T\mathbf{z} = -1, \quad \mathbf{uz} = \mathbf{0}, \quad \mathbf{u}, \mathbf{z} \geq \mathbf{0}. \tag{2}$$

As introduced by Cameron and Edmonds [1] an EP (Existentially Polynomial-time) theorem is a theorem of the form:

$$[\forall x : F_1(x), F_2(x), \ldots, F_k(x)],$$

where $F_i(x)$, $(i = 1, \ldots, k)$ is a predicate formula which has the form

$$F_i(x) = [\exists y_i \text{ such that } \|y_i\| \leq \|x\|^{n_i} \text{ and } f_i(x, y_i)].$$

Here $n_i \in \mathbb{Z}^+$, $\|z\|$ denotes the encoding length of z and $f_i(x, y_i)$ is a predicate for which there is a polynomial-size certificate.

The LCP duality theorem in EP form [6] is as follows:

[1] The definition of matrix classes is given in the next section.

Theorem 1 *Let matrix $M \in \mathbb{Q}^{n\times n}$ and vector $\mathbf{q} \in \mathbb{Q}^n$ be given. At least one of the following statements hold:*

(1) problem LCP has a complementary feasible solution $(\mathbf{x}, \mathbf{s})$, whose encoding size is polynomially bounded.
(2) problem $DLCP$ has a complementary feasible solution $(\mathbf{u}, \mathbf{z})$, whose encoding size is polynomially bounded.
(3) matrix M is not sufficient and there is a certificate whose encoding size is polynomially bounded.

The criss-cross algorithm for sufficient matrix LCPs was introduced by Hertog et al. [4]. The first criss-cross type pivot algorithm in EP form, which does not use apriori knowledge of sufficiency of the matrix M, was given by Fukuda et al. [6]. They utilized the LCP duality theorem of Fukuda and Terlaky [5]. Csizmadia and Illés [3] extended this method to several flexible finite pivot rules. These variants of the criss-cross method solve LCPs with an arbitrary matrix. They either solve the primal LCP or give a dual solution, or detect that the algorithm may begin cycling (due to lack of sufficiency) and in this case they give a polynomial size certificate of the lack of sufficiency. No interior point algorithm in EP form were published yet.

Summarizing, our aim is to construct interior point algorithms, that according to the duality theorem of LCP in EP form either give a solution of the original LCP or for the dual LCP, or detects the lack of property $\mathcal{P}_*(\tilde{\kappa})$, and gives a polynomial certificate in all cases in polynomial time.

The rest of the paper is organized as follows. The following section deals with the fundamental properties of $\mathcal{P}_*(\kappa)$-matrices and with some related results. In Section 3 we describe the general idea of modified IPM and than present the modified long-step path-following algorithm. Section 4 addresses the question, how the interior point assumption can be eliminated by using the embedding technique. For ease of understanding and to be self contained we collect the necessary results of [15] in the Appendix.

1.1 Notations

We use the following notations throughout the paper. Scalars and indices are denoted by lowercase Latin letters, vectors by lowercase boldface Latin letters, matrices by capital Latin letters, and finally sets by capital calligraphic letters. $\mathbb{R}^n_\oplus$ $(\mathbb{R}^n_+)$ is the nonnegative (positive) orthant of $\mathbb{R}^n$. Further, X is the diagonal matrix whose diagonal elements are the coordinates of the vector $\mathbf{x}$, so $X = \text{diag}(\mathbf{x})$, and I denotes the identity matrix of appropriate dimension. The vector $\mathbf{xs} = X\mathbf{s}$ is the componentwise product (Hadamard product) of the vectors $\mathbf{x}$ and $\mathbf{s}$, and for $\alpha \in \mathbb{R}$ the vector $\mathbf{x}^\alpha$ denotes the vector whose ith component is x_i^α. We denote the vector of ones by $\mathbf{e}$. Furthermore, for vector $\mathbf{x}$ we define the sets $\mathcal{I}_+(\mathbf{x}) = \{1 \le i \le n : x_i(Mx)_i > 0\}$ and $\mathcal{I}_-(\mathbf{x}) = \{1 \le i \le n : x_i(Mx)_i < 0\}$, which are used in the definition of $\mathcal{P}_*(\kappa)$ matrices.

Let the current point be $(\mathbf{x}, \mathbf{s})$ and $(\Delta\mathbf{x}, \Delta\mathbf{s})$ be the corresponding Newton direction.[2] The new point with step length θ is given by $(\mathbf{x}(\theta), \mathbf{s}(\theta)) = (\mathbf{x} + \theta\Delta\mathbf{x}, \mathbf{s} + \theta\Delta\mathbf{s})$. We use the δ_c centrality measures, where

$$\delta_c(\mathbf{xs}, \mu) = \left\| \sqrt{\frac{\mathbf{xs}}{\mu}} - \sqrt{\frac{\mu}{\mathbf{xs}}} \right\|.$$

[2] Generally the Newton direction is the unique solution of system (4), see page 333. We will discuss how to define the actual Newton direction for the long-step path-following algorithm in Section 3.

2 Matrix classes and the Newton step

The class of $P_*(\kappa)$-matrices were introduced by Kojima et al. [13], and it can be considered as a generalization of the class of positive semidefinite matrices (see Notations for definition of sets $\mathcal{I}_+, \mathcal{I}_-$).

Definition 2 Let $\kappa \geq 0$ be a nonnegative number. A matrix $M \in \mathbb{R}^{n\times n}$ is a $\mathcal{P}_*(\kappa)$-matrix if for all $\mathbf{x} \in \mathbb{R}^n$

$$(1+4\kappa) \sum_{i\in\mathcal{I}_+(\mathbf{x})} x_i(Mx)_i + \sum_{i\in\mathcal{I}_-(\mathbf{x})} x_i(Mx)_i \geq 0. \tag{3}$$

The nonnegative real number κ denotes the weight that need to be used at the positive terms so that the weighted 'scalar product' is nonnegative for each vector $\mathbf{x} \in \mathbb{R}^n$. Therefore, naturally $\mathcal{P}_*(0)$ is the class of positive semidefinite matrices (if we set aside the symmetry of the matrix M).

Definition 3 A matrix $M \in \mathbb{R}^{n\times n}$ is called a $\mathcal{P}_*$-matrix if it is a $\mathcal{P}_*(\kappa)$-matrix for some $\kappa \geq 0$, i.e.

$$\mathcal{P}_* = \bigcup_{\kappa\geq 0} \mathcal{P}_*(\kappa).$$

The class of sufficient matrices was introduced by Cottle et al. [2].

Definition 4 A matrix $M \in \mathbb{R}^{n\times n}$ is a column sufficient matrix if for all $\mathbf{x} \in \mathbb{R}^n$

$$X(M\mathbf{x}) \leq 0 \;\text{ implies }\; X(M\mathbf{x}) = 0,$$

and row sufficient if M^T is column sufficient. Matrix M is sufficient if it is both row and column sufficient.

Kojima et al. [13] proved that any $\mathcal{P}_*$-matrix is column sufficient and Guu and Cottle [7] proved that it is row sufficient, too. Therefore, each $\mathcal{P}_*$-matrix is sufficient. Väliaho proved the other direction of inclusion [19], thus the class of $\mathcal{P}_*$-matrices coincides with the class of sufficient matrices.

Definition 5 A matrix $M \in \mathbb{R}^{n\times n}$ is a $\mathcal{P}_0$-matrix, if all of its principal minors are nonnegative.

For further use we recall some results about $P_*(\kappa)$ and P_0 LCPs. The reader may consult the book of Kojima et al. [13, Lemma 4.1 p. 35] for the proof of the following Proposition.

Proposition 6 *A matrix $M \in \mathbb{R}^{n\times n}$ is a $\mathcal{P}_0$-matrix if and only if*

$$M' = \begin{bmatrix} -M & I \\ S & X \end{bmatrix} \text{ is a nonsingular matrix}$$

for any positive diagonal matrices $X, S \in \mathbb{R}^{n\times n}$.

Proposition 6 enables us to check whether matrix M is $\mathcal{P}_0$ or not. The next statement is used to guarantee the existence and uniqueness of Newton directions that are the solution of system (4) for various values of vector $\mathbf{a} \in \mathbb{R}^n$, where $\mathbf{a}$ depends on the particular interior point algorithm.

Corollary 7 *Let $M \in \mathbb{R}^{n\times n}$ be a $\mathcal{P}_0$-matrix, $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n_+$. Then for all $\mathbf{a} \in \mathbb{R}^n$ the system*

$$\begin{aligned} -M\Delta\mathbf{x} + \Delta\mathbf{s} &= \mathbf{0} \\ \mathbf{s}\Delta\mathbf{x} + \mathbf{x}\Delta\mathbf{s} &= \mathbf{a} \end{aligned} \tag{4}$$

has a unique solution $(\Delta\mathbf{x}, \Delta\mathbf{s})$.

The following estimations for the Newton direction are used in the complexity analysis of interior point methods. The next lemma is proved by Potra in [16].

Lemma 8 *Let M be an arbitrary $n \times n$ real matrix and $(\Delta\mathbf{x}, \Delta\mathbf{s})$ be a solution of system (4). Then*

$$\sum_{i\in\mathcal{I}_+} \Delta x_i \Delta s_i \le \frac{1}{4}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2.$$

Now we recall some inequalities, where the property of the matrix plays a crucial role.

Lemma 9 *Let matrix M be a $\mathcal{P}_*(\kappa)$-matrix and $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n_+$, $\mathbf{a} \in \mathbb{R}^n$. Let $(\Delta\mathbf{x}, \Delta\mathbf{s})$ be the solution of system (4). Then*

$$\|\Delta\mathbf{x}\Delta\mathbf{s}\|_\infty \le \left(\frac{1}{4}+\kappa\right)\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2, \quad \|\Delta\mathbf{x}\Delta\mathbf{s}\|_1 \le \left(\frac{1}{2}+\kappa\right)\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2,$$

$$\|\Delta\mathbf{x}\Delta\mathbf{s}\|_2 \le \sqrt{\left(\frac{1}{4}+\kappa\right)\left(\frac{1}{2}+\kappa\right)}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2.$$

The first statement's proof in the lemma is based on the $\mathcal{P}_*(\kappa)$ property of matrix M, and the proof technique is well known in from the literature, see e.g., the proof of Lemma 5.1 by Illés et al. [8]. The second estimation follows from the previous lemma by using some properties of $\mathcal{P}_*(\kappa)$-matrices, and the last estimation is a corollary of the first and second statements using some properties of norms.

3 Interior point algorithms in EP form

Our aim is to modify interior point algorithms in such a way, that they solve the LCP with any arbitrary matrix, or give a certificate, that the matrix is not $\mathcal{P}_*(\tilde{\kappa})$, where $\tilde{\kappa}$ is a given (arbitrary big) number. Potra and Liu [17] gave the first interior point algorithm, where we do not need to know apriori the value of κ, it is enough to know that the matrix is $\mathcal{P}_*$. Their algorithm initially assumes that the matrix is $\mathcal{P}_*(1)$. At each iteration they check whether the new point is in the appropriate neighborhood of the central path, or not. In the latter case they double the value of κ. We use this idea in a modified way. Because the larger κ is, the worse the iteration complexity is, we take only the necessary enlargement of κ (until it reaches $\tilde{\kappa}$). The inequality in the definition of $\mathcal{P}_*(\kappa)$-matrices gives the following lower bound on κ for any vector $\mathbf{x} \in \mathbb{R}^n$:

$$\kappa \ge \kappa(\mathbf{x}) = -\frac{1}{4}\frac{\mathbf{x}^T M\mathbf{x}}{\sum_{i\in\mathcal{I}_+} x_i (Mx)_i}.$$

In IPMs the $\mathcal{P}_*(\kappa)$ property need to be true only for the actual Newton direction $\Delta\mathbf{x}$ in various ways, for example this property ensures that with a certain step size the new iterate is in an appropriate neighborhood of the central path and/or the complementarity gap is sufficiently reduced. Consequently, if the desired results do not hold with the current κ value, we update κ to the lower bound determined by the Newton direction $\Delta\mathbf{x}$, i.e.,

$$\kappa(\Delta\mathbf{x}) = -\frac{1}{4}\frac{\Delta\mathbf{x}^T\Delta\mathbf{s}}{\sum_{i\in\mathcal{I}_+}\Delta x_i\Delta s_i} \qquad (\Delta\mathbf{s} = M\Delta\mathbf{x}). \tag{5}$$

The following two lemmas are immediate consequences of the definition of $\mathcal{P}_*(\kappa)$ and $\mathcal{P}_*$-matrices.

Lemma 10 *Let M be a real $n\times n$ matrix. If there exists a vector $\mathbf{x}\in\mathbb{R}^n$ such that $\kappa(\mathbf{x}) > \tilde{\kappa}$, then the matrix M is not $\mathcal{P}_*(\tilde{\kappa})$ and $\mathbf{x}$ is a certificate for this fact.*

Lemma 11 *Let M be a real $n\times n$ matrix. If there exists a vector $\mathbf{x}\in\mathbb{R}^n$ such that $\mathcal{I}_+(\mathbf{x}) = \{i\in\mathcal{I} : x_i(Mx)_i > 0\} = \emptyset$, then the matrix M is not $\mathcal{P}_*$ and $\mathbf{x}$ is a certificate for this fact.*

Therefore, if there exists such a vector $\Delta\mathbf{x}$ for which $\mathcal{I}_+ = \emptyset$, and thus $\kappa(\Delta\mathbf{x})$ is not defined, then the matrix M of the LCP is not a $\mathcal{P}_*$-matrix. In this case we stop the algorithm, and the output will be $\Delta\mathbf{x}$ as a certificate to prove that M is not a $\mathcal{P}_*$-matrix.

There is another point where IPMs may fail if the matrix of the LCP is not $\mathcal{P}_*$. If the matrix is not $\mathcal{P}_0$, then the Newton system may not have a solution, or the solution may not be unique (see Corollary 7). If this is the case, then the actual point $(\mathbf{x},\mathbf{s})$ is a certificate which proves that the matrix is not $\mathcal{P}_0$, so it is not $\mathcal{P}_*$ either.

Summarizing, we make three tests in our algorithms. In each tests the property of the LCP matrix M is examined indirectly. When we inquire about the existence and uniqueness of the solution of the Newton system, we check whether the matrix is $\mathcal{P}_0$, or not. When we test some properties of the new point, for example whether it is in the appropriate neighborhood of the central path, we examine the $\mathcal{P}_*(\kappa)$ property for the current value of κ. Finally, if the $\kappa(\Delta\mathbf{x})$ value given by (5) is not defined, then the matrix is not $\mathcal{P}_*$. We note that at each step all properties are checked only locally, only for one vector of $\mathbb{R}^n$. Consequently, it is possible, that the matrix is not a $\mathcal{P}_0$ or $\mathcal{P}_*$-matrix, but the algorithm does not discover it and solves the LCP in polynomial time, because those properties were true for the vectors $\mathbf{x}$ and $\Delta\mathbf{x}$ that were generated by the algorithm. It may also occur, that the matrix is not $\mathcal{P}_*$, but the algorithm does not detect it. It only increases the value of κ if $\kappa < \kappa(\Delta\mathbf{x})$ and then it proceeds to the next iterate. This is the reason why we need the threshold $\tilde{\kappa}$ parameter that enables us to get a finite algorithm. In practice this is not a real restriction, because for big values of κ the step length might be smaller than the machine precision.

The following lemma is our main tool to verify when the $\mathcal{P}_*(\kappa)$ property does not hold. Furthermore, the concerned vector $\Delta\mathbf{x}$ is a certificate, whose encoding size is polynomial when it is computed as the solution of the Newton system (4). We use this lemma during the analysis. The first statement is simply the negation of the definition. We point out in Lemma 13 that if Theorem 10.5 of [15] does not hold, then the second or the third statement is realized.

Lemma 12 *If one of the following statements holds then the matrix M is not a $\mathcal{P}_*(\kappa)$-matrix.*

1. *There exists a vector $\mathbf{y}\in\mathbb{R}^n$ such that*

$$(1+4\kappa)\sum_{i\in\mathcal{I}_+(\mathbf{y})} y_i w_i + \sum_{i\in\mathcal{I}_-(\mathbf{y})} y_i w_i < 0,$$

where $\mathbf{w} = M\mathbf{y}$ and
$\mathcal{I}_+(\mathbf{y}) = \{i\in I : y_i w_i > 0\}$, $\mathcal{I}_-(\mathbf{y}) = \{i\in I : y_i w_i < 0\}$.

2. *There exists a solution* $(\Delta\mathbf{x}, \Delta\mathbf{s})$ *of system (4) such that*

$$\|\Delta\mathbf{x}\Delta\mathbf{s}\|_\infty > \frac{1+4\kappa}{4}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2.$$

3. *There exists a solution* $(\Delta\mathbf{x}, \Delta\mathbf{s})$ *of system (4) such that*

$$\max\left(\sum_{i\in\mathcal{I}_+}\Delta x_i\Delta s_i,\ -\sum_{i\in\mathcal{I}_-}\Delta x_i\Delta s_i\right) > \frac{1+4\kappa}{4}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2.$$

Proof The first statement is the negation of the definition of $\mathcal{P}_*(\kappa)$ matrices. Now we prove that the first statement follows from the others. By Lemma 8, one has

$$\sum_{i\in\mathcal{I}_+}\Delta x_i\Delta s_i \le \frac{1}{4}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2, \tag{6}$$

so $\Delta x_i\Delta s_i \le \|\mathbf{a}/\sqrt{\mathbf{xs}}\|^2/4$ for all $i\in\mathcal{I}_+$. Accordingly, if the inequality of the second statement holds, let $j\in\mathcal{I}$ such that $\|\Delta\mathbf{x}\Delta\mathbf{s}\|_\infty = |\Delta x_j\Delta s_j|$, then $j\in\mathcal{I}_-$, i.e., $\Delta x_j\Delta s_j < 0$. Therefore

$$\begin{aligned}(1+4\kappa)\sum_{i\in\mathcal{I}_+}\Delta x_i\Delta s_i + \sum_{i\in\mathcal{I}_-}\Delta x_i\Delta s_i &\le (1+4\kappa)\sum_{i\in\mathcal{I}_+}\Delta x_i\Delta s_i + \Delta x_j\Delta s_j\\ &< (1+4\kappa)\frac{1}{4}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2 - \frac{1+4\kappa}{4}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2 = 0.\end{aligned} \tag{7}$$

This is the same as the first statement with $\mathbf{y}=\Delta\mathbf{x},\ \mathbf{w}=\Delta\mathbf{s}$.

From the assumption of statement 3 and inequality (6), the second term is greater in the maximum, hence one has

$$\sum_{i\in\mathcal{I}_-}\Delta x_i\Delta s_i < -\frac{1+4\kappa}{4}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2.$$

Therefore $(\Delta\mathbf{x}, \Delta\mathbf{s})$ satisfies inequality (7), so $\mathbf{y}=\Delta\mathbf{x},\ \mathbf{w}=\Delta\mathbf{s}$ proves that the first statement holds.

The proof of the last statement, by using inequality (6) follows from the following inequality

$$\begin{aligned}(1+4\kappa)\sum_{i\in\mathcal{I}_+}\Delta x_i\Delta s_i + \sum_{i\in\mathcal{I}_-}\Delta x_i\Delta s_i &= 4\kappa\sum_{i\in\mathcal{I}_+}\Delta x_i\Delta s_i + \sum_{i\in I}\Delta x_i\Delta s_i\\ &< \kappa\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2 - \kappa\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2 = 0,\end{aligned}$$

where we can use $\mathbf{y}=\Delta\mathbf{x},\ \mathbf{w}=\Delta\mathbf{s}$ again to get the first statement. □

3.1 Long-step path-following interior point algorithm

In this section we deal with the algorithm proposed in [15]. The long-step algorithm has two loops. In the inner loop one take steps towards the central path and in the outer loop the parameter μ is updated. In this algorithm we check the decrease of the centrality measure after one inner step and if it is too small, then κ is updated by (5), or a certificate is obtained showing that M is not a $\mathcal{P}_*(\tilde{\kappa})$ matrix. As stated in the previous subsection, if $\kappa(\Delta\mathbf{x})$ is not defined, then the matrix is not $\mathcal{P}_*$ and $\Delta\mathbf{x}$ is a certificate of it. Furthermore if $\kappa(\Delta\mathbf{x}) > \tilde{\kappa}$, then matrix M is not $\mathcal{P}_*(\tilde{\kappa})$ and the Newton direction $\Delta\mathbf{x}$ is a certificate for this fact. The modified algorithm is as follows:

Long-step path-following interior point algorithm

Input:

an upper bound $\tilde{\kappa} > 0$ on the value of κ;
a proximity parameter $\tau \geq 2$;
an accuracy parameter $\varepsilon > 0$;
a fix barrier update parameter $\gamma \in (0, 1)$;
an initial point $(\mathbf{x}^0, \mathbf{s}^0)$, and $\mu^0 > 0$
such that $\delta_c(\mathbf{x}^0\mathbf{s}^0, \mu^0) = \left\| \sqrt{\frac{\mathbf{x^0s^0}}{\mu^0}} - \sqrt{\frac{\mu^0}{\mathbf{x^0s^0}}} \right\| < \tau.$

begin
  $\mathbf{x} := \mathbf{x}^0$, $\mathbf{s} := \mathbf{s}^0$, $\mu := \mu^0$, $\kappa := 0$;
  while $\mathbf{x}^T\mathbf{s} \geq \varepsilon$ **do**
    $\mu = (1-\gamma)\mu$;
    while $\delta_c(\mathbf{xs}, \mu) \geq \tau$ **do**
      calculate the Newton direction $(\Delta\mathbf{x}, \Delta\mathbf{s})$ with $\mathbf{a} = \mu\mathbf{e} - \mathbf{xs}$;
      if (the Newton direction does not exist or it is not unique) **then**
        return the matrix is not $\mathcal{P}_0$; *% see Corollary 7*
      $\bar{\theta} = \text{argmin}\,\{\delta_c(\mathbf{x}(\theta)\mathbf{s}(\theta), \mu) : \ (\mathbf{x}(\theta), \mathbf{s}(\theta)) > \mathbf{0}\}$;
      if $\left(\delta_c^2(\mathbf{xs}, \mu) - \delta_c^2(\mathbf{x}(\bar{\theta})\mathbf{s}(\bar{\theta}), \mu) < \frac{5}{3(1+4\kappa)}\right)$ **then**
        calculate $\kappa(\Delta\mathbf{x})$; *% see (5)*
        if =($\kappa(\Delta\mathbf{x})$ is not defined) **then**
          return the matrix is not $\mathcal{P}_*$; *% see Lemma 11*
        if $(\kappa(\Delta\mathbf{x}) > \tilde{\kappa})$ **then**
          return the matrix is not $\mathcal{P}_*(\tilde{\kappa})$; *% see Lemma 10*
        $\kappa = \kappa(\Delta\mathbf{x})$;
      $\mathbf{x} = \mathbf{x}(\bar{\theta}), \ \mathbf{s} = \mathbf{s}(\bar{\theta})$;
    end
  end
end.

We use the notations of [15]:

$$\sigma_+ = \frac{1}{\mu}\sum_{i\in\mathcal{I}_+}\Delta x_i\Delta s_i, \qquad \sigma_- = -\frac{1}{\mu}\sum_{i\in\mathcal{I}_-}\Delta x_i\Delta s_i, \qquad \sigma = \max(\sigma_+, \sigma_-).$$

Further, let

$$\theta_\ell^* := \frac{2}{(1+4\kappa)\delta_c^2(\mathbf{xs}, \mu)}.$$

To simplify the notation we write δ and δ^* instead of $\delta_c(\mathbf{xs}, \mu)$, $\delta_c(\mathbf{x}(\theta_\ell^*)\mathbf{x}(\theta_\ell^*), \mu)$, respectively.

Peng et al. [15] proved, that for $\mathcal{P}_*(\kappa)LCP$'s the step length θ_ℓ^* is feasible, and taking this step the decrease of the proximity measure is sufficient to ensure polynomiality of the algorithm. The following lemma shows that if the specified sufficiently large decrease does not take place, then the matrix of the problem is not $\mathcal{P}_*(\kappa)$.

Lemma 13 *If after an inner iteration the decrease of the proximity is not sufficient, i.e.,* $\delta^2(\mathbf{xs}, \mu) - \delta^2(\mathbf{x}(\bar{\theta})\mathbf{s}(\bar{\theta}), \mu) < \frac{5}{3(1+4\kappa)}$, *then the matrix of the LCP is not* $\mathcal{P}_*(\kappa)$ *with the actual* κ *value, and the Newton direction* $\Delta\mathbf{x}$ *is a certificate for this fact.*

Proof By Theorem 22 (see the Appendix), if the matrix is $\mathcal{P}_*(\kappa)$ we achieve the sufficient decrease of the centrality measure with step length θ_ℓ^*. Therefore, if the maximum decrease is smaller, then either $(\mathbf{x}^*, \mathbf{s}^*)$ is not feasible or the decrease of the proximity with step size θ_ℓ^* is not sufficient, i.e., $\delta^2(\mathbf{xs}, \mu) - \delta^2(\mathbf{x}^*\mathbf{s}^*, \mu) < \frac{5}{3(1+4\kappa)}$. We prove in both cases that the matrix of the problem is not $\mathcal{P}_*(\kappa)$ and $\Delta\mathbf{x}$ is a certificate of it.

If the point $(\mathbf{x}^*, \mathbf{s}^*)$ is not feasible, then $\mu\mathbf{e} + \theta_\ell^*\Delta\mathbf{x}\Delta\mathbf{s} \not> 0$, so there exists such an index k, that $\mu + \theta_\ell^*\Delta x_k\Delta s_k \le 0$. It means, that $\Delta x_k\Delta s_k \le -\mu/\theta_\ell^* = -\frac{\mu}{2}(1+4\kappa)\delta^2 < 0$. Since $(\Delta\mathbf{x}, \Delta\mathbf{s})$ is a solution of system (4) with $\mathbf{a} = \mu\mathbf{e} - \mathbf{xs}$, therefore

$$\|\Delta\mathbf{x}\Delta\mathbf{s}\|_\infty > \frac{1+4\kappa}{4}\mu\delta^2 = \frac{1+4\kappa}{4}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2,$$

but this contradicts to the $\mathcal{P}_*(\kappa)$ property by the second statement of Lemma 12.

Now let us analyze the case when the decrease of the proximity measure is not sufficient with step length θ_ℓ^*. According to the condition of Theorem 21 (see the Appendix), let us consider the cases $\theta_\ell^* < 1/\sigma$ and $\theta_\ell^* \ge 1/\sigma$ separately. If $\theta_\ell^* < 1/\sigma$, by the definition of θ_ℓ^* and Theorem 21, one has

$$(\delta^*)^2 - \delta^2 \le -\frac{2}{1+4\kappa} + \frac{2(\theta_\ell^*)^3\sigma^2}{1-(\theta_\ell^*)^2\sigma^2}. \tag{8}$$

Since $\delta \ge 2$, we can write

$$-\frac{2}{1+4\kappa} + \frac{4}{3(1+4\kappa)\delta^2} \le -\frac{5}{3(1+4\kappa)}. \tag{9}$$

Therefore, by inequalities (8) and (9) and by the assumption of the lemma, the following inequalities hold

$$-\frac{2}{1+4\kappa} + \frac{4}{3(1+4\kappa)\delta^2} \le -\frac{5}{3(1+4\kappa)} < (\delta^*)^2 - \delta^2 \le -\frac{2}{1+4\kappa} + \frac{2(\theta_\ell^*)^3\sigma^2}{1-(\theta_\ell^*)^2\sigma^2}.$$

Using the definition of θ_ℓ^* we get

$$\frac{4}{3(1+4\kappa)\delta^2} < \frac{2(\theta_\ell^*)^2\sigma^2}{1-(\theta_\ell^*)^2\sigma^2}\,\frac{2}{(1+4\kappa)\delta^2}.$$

After reordering one has $\frac{1}{2} < \theta_\ell^*\sigma$. Substituting the definition of θ_ℓ^* we get the following lower bound on σ

$$\max(\sigma_+, \sigma_-) = \sigma > \frac{1+4\kappa}{4}\delta^2. \tag{10}$$

By the definitions of σ and the proximity measure, one has

$$\max\left(\sum_{i\in\mathcal{I}_+}\Delta x_i\Delta s_i, -\sum_{i\in\mathcal{I}_-}\Delta x_i\Delta s_i\right) > \frac{1+4\kappa}{4}\mu\delta^2 = \frac{1+4\kappa}{4}\left\|\frac{\mu\mathbf{e}-\mathbf{xs}}{\sqrt{\mathbf{xs}}}\right\|^2.$$

By the third statement of Lemma 12 this implies that matrix M is not $\mathcal{P}_*(\kappa)$ and the vector $\Delta\mathbf{x}$ is a certificate of it.

If $\theta_\ell^* \geq 1/\sigma$, then by the definition of θ_ℓ^* one has $\sigma \geq (1+4\kappa)\delta^2/2$, therefore inequality (10) holds, so the lemma is true in this case, too. □

The following lemma proves, that the long-step path-following IPM is well defined.

Lemma 14 *At each iteration when the value of κ is updated, then the new value of κ satisfies the inequality $\delta^2(\mathbf{xs},\mu) - \delta^2(\mathbf{x}(\bar\theta)\mathbf{s}(\bar\theta),\mu) \geq \frac{5}{3(1+4\kappa)}$.*

Proof In the proof of Theorem 22 we use the $\mathcal{P}_*(\kappa)$ property only for the vector $\Delta\mathbf{x}$. When parameter κ is updated, then we choose the new value so that the inequality in the definition of $\mathcal{P}_*(\kappa)$-matrices (3) holds for the vector $\Delta\mathbf{x}$. Therefore the new point defined by the updated value of step size θ_ℓ^* is strictly feasible and $\delta^2(\mathbf{xs},\mu)-\delta^2(\mathbf{x}^*\mathbf{s}^*,\mu)) \geq \frac{5}{3(1+4\kappa)}$. Thus the new value of θ_ℓ^* was considered in the definition of $\bar\theta$ as $\delta^2(\mathbf{xs},\mu) - \delta^2(\mathbf{x}(\bar\theta)\mathbf{s}(\bar\theta),\mu) \geq \delta^2(\mathbf{xs},\mu) - \delta^2(\mathbf{x}^*\mathbf{s}^*,\mu) \geq \frac{5}{3(1+4\kappa)}$. □

Now we are ready to state the complexity result for the modified long-step path-following interior point algorithm for general LCP in case an initial interior point is given.

Theorem 15 *Let $\tau = 2$, $\gamma = 1/2$ and $(\mathbf{x}^0,\mathbf{s}^0)$ be a feasible interior point such that $\delta_c(\mathbf{x}^0\mathbf{s}^0,\mu^0) \leq \tau$. Then after at most $\mathcal{O}\left((1+4\hat\kappa)n\log\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon}\right)$ steps, where $\hat\kappa \leq \tilde\kappa$ is the largest value of parameter κ throughout the algorithm, the long-step path-following interior point algorithm either produces a point $(\hat{\mathbf{x}},\hat{\mathbf{s}})$ such that $\hat{\mathbf{x}}^T\hat{\mathbf{s}} \leq \varepsilon$ and $\delta_c(\hat{\mathbf{x}}\hat{\mathbf{s}},\hat\mu) \leq \tau$ or it gives a certificate that the matrix of the LCP is not $\mathcal{P}_*(\tilde\kappa)$.*

Proof The algorithm at each iteration either takes a step, or detects, that the matrix is not $\mathcal{P}_*(\tilde\kappa)$ and stops. If we take a Newton step, then by the definition of the algorithm and by Lemma 14 the decrease of the squared proximity measure is at least $5/[3(1+4\kappa)]$. We can see, that larger κ means smaller lower bound on decrease of the proximity measure. Therefore, if the algorithm stops with an ε-optimal solution, then after each Newton step the decrease of the squared proximity measure is at least $5/[3(1+4\hat\kappa)]$. Thus at each outer iteration we take at most as many inner iterations as in the original long-step algorithm with a $\mathcal{P}_*(\hat\kappa)$-matrix do, or the algorithm stops earlier with a certificate that M is not $\mathcal{P}_*(\tilde\kappa)$-matrix. By the complexity theorem of the original algorithm (see Theorem 23 in the Appendix) we proved our statement. □

3.2 An EP theorem for LCPs based on interior point algorithms

It is known from the literature [13] that assuming $\mathcal{F}^0 \neq \emptyset$ and the matrix of the LCP is sufficient, then the LCP has a solution. According to this result, by making use of the complexity theorem of the previous section (Theorem 15), and the rounding procedure of [9] we can now present the following EP type theorem. We assume that the data are rational (solving problems with computer this is a reasonable assumption), ensuring polynomial encoding size of certificates and polynomial complexity of the algorithm.

Theorem 16 *Let an arbitrary matrix $M \in \mathbb{Q}^{n\times n}$, a vector $\mathbf{q} \in \mathbb{Q}^n$ and a point $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{F}^0$ with $\delta_c(\mathbf{x}^0\mathbf{s}^0, \mu^0) \leq \tau$ be given. Then one can verify in polynomial time that at least one of the following statements holds*

(1) problem LCP has a feasible complementary solution $(\mathbf{x}, \mathbf{s})$ whose encoding size is polynomially bounded.
(2) matrix M is not in the class of $\mathcal{P}_*(\tilde{\kappa})$ and there is a certificate whose encoding size is polynomially bounded.

4 Solving general LCPs without having an initial interior point

When for an LCP no initial interior point is known then we have two possibilities: (i) apply an infeasible interior point algorithm, or (ii) use the embedding technique of Kojima et al. [13]. In this section we discuss the solution method based on the embedding technique.

4.1 Embedded model for general LCPs

In this section we deal with a technique that allows us to handle the initialization problem of IPMs for LCPs, i.e., how to get a well centered initial interior point. The embedding model discussed in this section was introduced by Kojima et al. [13]. The following lemma plays a crucial role in this model.

Lemma 17 (Lemma 5.3 in [13]) *Let M be a real matrix. The matrix $M' = \begin{pmatrix} M & I \\ -I & O \end{pmatrix}$ belongs to the class $\mathcal{P}_0$, column sufficient, $\mathcal{P}_*$, $\mathcal{P}_*(\kappa)$, positive semidefinite or skew symmetric if and only if M belongs to the same matrix class.*

Let us consider the LCP as given by (1). We assume that all the entries of matrix M and vector $\mathbf{q}$ are integral. The input length L of problem LCP is defined as

$$L = \sum_{i=1}^{n}\sum_{j=1}^{n} \log_2(|m_{ij} + 1|) + \sum_{i=1}^{n} \log_2(|q_i| + 1) + 2\log_2 n,$$

and let

$$\tilde{\mathbf{q}} = \frac{2^{L+1}}{n^2}\,\mathbf{e}.$$

The embedding problem of Kojima et al. [13] is as follows:

$$\left.\begin{aligned} -M'\mathbf{x}' + \mathbf{s}' &= \mathbf{q}' \\ \mathbf{x}'\mathbf{s}' &= \mathbf{0} \\ \mathbf{x}', \mathbf{s}' &\geq \mathbf{0} \end{aligned}\right\} \qquad (LCP')$$

where

$$\mathbf{x}' = \begin{pmatrix} \mathbf{x} \\ \tilde{\mathbf{x}} \end{pmatrix}, \; \mathbf{s}' = \begin{pmatrix} \mathbf{s} \\ \tilde{\mathbf{s}} \end{pmatrix}, \; \mathbf{q}' = \begin{pmatrix} \mathbf{q} \\ \tilde{\mathbf{q}} \end{pmatrix}, \; M' = \begin{pmatrix} M & I \\ -I & O \end{pmatrix}.$$

An initial interior point for the embedding model (LCP') is readily available:

$$\mathbf{x} = \frac{2^L}{n^2}\mathbf{e}, \quad \tilde{\mathbf{x}} = \frac{2^{2L}}{n^3}\mathbf{e}, \quad \mathbf{s} = \frac{2^L}{n^2}M\mathbf{e} + \frac{2^{2L}}{n^3}\mathbf{e} + \mathbf{q}, \quad \tilde{\mathbf{s}} = \frac{2^L}{n^2}\mathbf{e}.$$

The following lemma indicates the connection between the solutions of the embedding problem and the solutions of the original LCP.

Lemma 18 **(Lemma 5.4 in [13])** *Let* $(\mathbf{x}', \mathbf{s}') = \left(\begin{pmatrix}\mathbf{x}\\ \tilde{\mathbf{x}}\end{pmatrix}, \begin{pmatrix}\mathbf{s}\\ \tilde{\mathbf{s}}\end{pmatrix}\right)$ *be a solution of problem* (LCP').

(i) If $\tilde{\mathbf{x}} = \mathbf{0}$*, then* $(\mathbf{x}, \mathbf{s})$ *is a solution of the original* LCP.
(ii) If M *is column sufficient and* $\tilde{\mathbf{x}} \neq \mathbf{0}$*, then the original* LCP *has no solution.*

4.2 Using dual information

We deal with the dual of the LCP in our paper [10]. Let us denote the set of the dual feasible solutions by $\mathcal{F}_D := \left\{(\mathbf{u}, \mathbf{z}) \geq \mathbf{0} : \ \mathbf{u} + M^T\mathbf{z} = \mathbf{0}, \ \mathbf{q}^T\mathbf{z} = -1\right\}$. The following result is proved:

Lemma 19 *Let matrix* M *be row sufficient. If* $(\mathbf{u}, \mathbf{z}) \in \mathcal{F}_D$*, then* $(\mathbf{u}, \mathbf{z})$ *is a solution of* $DLCP$.

Based on Lemma 19 let us approach the problem from the dual side. First try to solve the feasibility problem of $DLCP$. It is a linear optimization problem, therefore we can solve it in polynomial time. We have the following cases:

(a) $\mathcal{F}_D \neq \emptyset$ and $\mathbf{uz} = \mathbf{0}$ holds for the computed $(\mathbf{u}, \mathbf{z}) \in \mathcal{F}_D$: then we solved problem $DLCP$.
(b) $\mathcal{F}_D \neq \emptyset$ and for the computed $(\mathbf{u}, \mathbf{z}) \in \mathcal{F}_D$: $\mathbf{uz} \neq \mathbf{0}$ holds, then by Lemma 19 we know that M is not a row sufficient matrix, therefore it is not a sufficient matrix either, and vector $\mathbf{z}$ is a certificate for this.
(c) $\mathcal{F}_D = \emptyset$ then problem $DLCP$ has no solution.

In cases (a) and (b) we have solved the LCP in the sense of Theorem 1. In case (c) we try to solve the embedded problem (LCP') using the modified path-following algorithm. The modified algorithm either shows that matrix M', and thus by Lemma 18 matrix M as well, is not in the class of $\mathcal{P}_*(\tilde{\kappa})$ or solves problem (LCP'). In the latter case we have two subcases:

(i) $\tilde{\mathbf{x}} = \mathbf{0}$ then by Lemma 18 LCP has a solution.
(ii) $\tilde{\mathbf{x}} \neq \mathbf{0}$.

When $\tilde{\mathbf{x}} \neq \mathbf{0}$ and $\mathcal{F}_D = \emptyset$, then if matrix M is sufficient, then it is also column sufficient so the LCP has no solution by Lemma 18. But this contradicts to the Fukuda-Terlaky LCP duality theorem [3,5,6], therefore in this case matrix M cannot be sufficient and the vector $\tilde{\mathbf{x}}$ is an indirect certificate for this.

The dual side approach combining with the complexity result Theorem 16 (an interior point of problem (LCP') is known by construction) we can state our main result.

Theorem 20 *Let an arbitrary matrix* $M \in \mathbb{Q}^{n\times n}$ *and a vector* $\mathbf{q} \in \mathbb{Q}^n$ *be given. Then one can verify in polynomial time that at least one of the following statements hold*

(1) the LCP *problem (1) has a feasible complementary solution* $(\mathbf{x}, \mathbf{s})$ *whose encoding size is polynomially bounded.*
(2) problem $DLCP$ *has a feasible complementary solution* $(\mathbf{u}, \mathbf{z})$ *whose encoding size is polynomially bounded.*
(3) matrix M *is not in the class* $\mathcal{P}_*(\tilde{\kappa})$.

Theorem 20 is a generalization of Theorem 16. Since the interior point assumption is eliminated, it can occur that the LCP has no solution while matrix M is sufficient. This is the second statement of Theorem 20. On the other hand, as we have seen in (see page 340) case

(ii) in the dual side approach, when the matrix is not sufficient, but we have only an indirect certificate $\tilde{\mathbf{x}}$. This is the reason why in the last case of Theorem 20 we cannot ensure an explicit certificate. Therefore, Theorem 20 is stronger than Theorem 16, because the interior point assumption is eliminated, however, only an indirect certificate is provided in the last case.

It is interesting to note that Theorem 20 and Theorem 1 (a result of Fukuda et al. [6]) are different in two aspects: first, our statement (3) is weaker in some cases then theirs (there is no direct certificate in one case), but on the other hand our constructive proof is based on polynomial time algorithms and a polynomial size certificate is provided in all other cases in polynomial time.

5 Summary

Cameron and Edmonds' [1] EP theorem and its LCP form proven by Fukuda et al. [6] motivated our research. The use of the LCP-duality theorem (Theorem 1) in EP form is a novel idea in the interior point literature.

Among others, Potra and Liu [17] extended some IPMs for sufficient matrix LCPs. Our aim was to modify IPMs in such a way that they may be applied to LCPs without any restriction or knowledge about the properties of the coefficient matrices. We have shown that LCPs with arbitrary matrix can be solved in polynomial time in the following extended manner: in polynomial time we either solve the problem up to ε-optimality or show that the matrix does not belong to the class of $\mathcal{P}_*(\tilde{\kappa})$ matrices, for which a polynomial size certificate is provided by the algorithm.

Appendix

To make our paper self contained, here we present those results from [15] that are needed for our developments. All theorems are converted to our notation.

Theorem 21 (Theorem 10.2 in [15]) *Let M be an arbitrary matrix, $(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^0$, $(\Delta\mathbf{x}, \Delta\mathbf{s})$ is the Newton direction of the long-step path-following algorithm, $\delta := \delta_c(\mathbf{xs}, \mu)$ and $\delta^+ := \delta_c(\mathbf{x}(\theta)\mathbf{s}(\theta), \mu) \leq \tau$. Then for all $0 \leq \theta \leq 1/\sigma$, one has*

$$\delta^+ \leq (1-\theta)\delta^2 + \frac{2\theta^3\sigma^2}{1-\theta^2\sigma^2}.$$

Theorem 22 (Theorem 10.5 in [15]) *If $M \in \mathcal{P}_*(\kappa)$, $(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^0$, $(\Delta\mathbf{x}, \Delta\mathbf{s})$ is the Newton direction of the long-step path-following algorithm, $\delta := \delta_c(\mathbf{xs}, \mu) \geq 2$ and $\delta^* := \delta_c(\mathbf{x}(\theta^*)\mathbf{s}(\theta^*), \mu) \leq \tau$, where $\theta^* = \frac{2}{(1+4\kappa)\delta^2}$. Then*

$$(\delta^*)^2 - \delta^2 \leq -\frac{5}{3(1+4\kappa)}. \tag{11}$$

Theorem 23 (From Theorem 10.10 and the subsequent remarks in [15])
Let matrix $M \in \mathcal{P}_*(\kappa)$, $\tau = 2$, $\gamma = 1/2$ and $(\mathbf{x}^0, \mathbf{s}^0)$ be a feasible interior point such that $\delta_c(\mathbf{x}^0\mathbf{s}^0, \mu^0) \leq \tau$. Then the long-step path-following algorithm produces a point $(\hat{\mathbf{x}}, \hat{\mathbf{s}})$ such that $\delta_c(\hat{\mathbf{x}}\hat{\mathbf{s}}, \hat{\mu}) \leq \tau$ and $\hat{\mathbf{x}}^T\hat{\mathbf{s}} \leq \varepsilon$ in at most

$$\mathcal{O}\left((1+4\kappa)n \log \frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon}\right) \quad \textit{iterations.}$$

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